HERMAN RINGS OF MEROMORPHIC MAPS WITH AN OMITTED VALUE

TARAKANTA NAYAK

Abstract. Following are proved on Herman rings of meromorphic functions having at least one omitted value. If all the poles of such a function are multiple then it has no Herman ring. Herman ring of period one or two does not exist. Functions with a single pole or with at least two poles one of which is an omitted value have no Herman ring. Every doubly connected periodic Fatou component is a Herman ring.

1. Introduction

Unlike rational maps, a transcendental meromorphic map can omit a point of the Reimann sphere. Such a point is called an omitted value of the map and by Picard's theorem, there can be at most two such values. These values are known to be asymptotic values i.e., those to which the map $f(z)$ approaches when $z \to \infty$ along some curve. Not every asymptotic value is omitted.

A singular value of a transcendental meromorphic map f is either a critical value (the image of a point z for which $f'(z) = 0$) or an asymptotic value. Such a value is also known as a singularity of the inverse function f^{-1} because this is a point where at least one branch of f^{-1} fails to be defined. Further, there are different possible ways in which this failure can take place leading to the following classification of singularities [3]. For $a \in \widehat{\mathbb{C}}$ and $r > 0$, let $D_r(a)$ be a disk (in the spherical metric) and choose a component U_r of $f^{-1}(D_r(a))$ in such a way that $U_{r_1} \subset U_{r_2}$ for $0 < r_1 < r_2$. There are two possibilities.

(1) $\bigcap_{r>0} U_r = \{z\}$ for $z \in \mathbb{C}$: In this case, $f(z) = a$. The point z is called an *ordinary point* if (i)$f'(z) \neq 0$ and $a \in \mathbb{C}$, or (ii) z is a simple pole. The point z is called a *critical point* if $f'(z) = 0$ and $a \in \mathbb{C}$, or z is a multiple pole. In this case, a is called a *critical value* and we say that a critical point/algebraic singularity lies over a.

(2) $\bigcap_{r>0} U_r = \emptyset$: The choice $r \to U_r$ defines a transcendental singularity of f^{-1}. We say a singularity U lies over a. The singularity U lying over a is called direct if there exists $r > 0$ such that $f(z) \neq a$ for all $z \in U_r$. Otherwise it is called indirect.

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Over each asymptotic value, there lies a transcendental singularity and there is always a critical point lying over a critical value. It is important to note that an asymptotic value can also be a critical value. But an omitted value can neither be a critical value nor the image of any ordinary point. Further, each singularity lying over an omitted value is direct. In this way, an omitted value can be viewed as the simplest instance of a transcendental singularity.

A transcendental meromorphic function (for which ∞ is the only essential singularity) can (1) be entire, (2) be analytic self-map of the punctured plane i.e., with only one pole which is an omitted value or (3) have at least two poles or exactly one pole which is not an omitted value. The functions in the last category are usually referred as general meromorphic functions possibly because the property of being meromorphic has the clearest manifestation, at least in dynamical terms in this case. Let M denote the class of all general meromorphic maps and

$$M_o = \{f \in M \ : \ f \text{ has at least an omitted value}\}.$$

We deal with the functions belonging to the class M_o in this article.

For a meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the set of points $z \in \widehat{\mathbb{C}}$ in a neighbourhood of which the sequence of iterates $\{f^n(z)\}_{n=0}^{\infty}$ is defined and forms a normal family is called the Fatou set of f. The Julia set is its complement in $\widehat{\mathbb{C}}$. The Fatou set is open by definition and each of its maximally connected subset is known as a Fatou component. A Fatou component U is called p-periodic if p is the smallest natural number satisfying $f^p(U) \subseteq U$. Periodic Fatou components are of five types, namely Attracting domain, Parabolic domain, Siegel disk, Herman ring and Baker domain. A Herman ring H with period p is such that there exists an analytic homeomorphism $\phi : H \to A = \{z : 1 < |z| < r\}, r > 1$ with $\phi(f^p(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for all $z \in A$ and for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. clearly, there are uncountably many f^p-invariant Jordan curves in H. Each such curve separates the two components of $\widehat{\mathbb{C}} \setminus H$.

Transcendental entire functions always omit ∞ and at most another point in the plane. Analytic self-maps of the punctured plane omit two values, namely the pole and ∞. The forward orbit of an omitted value is either empty or finite in these cases. Though the forward orbit of an omitted value of a function in M_o can be infinite, functions belonging to M_o can be viewed as generalizations of above functions. It is well-known that entire functions and analytic self-maps of the punctured plane cannot have any Herman ring, the proof of the later appearing in [8]. Investigations on the role of omitted values in determining certain aspects of the dynamics of a function is initiated in [7]. It is seen, among other things, that in most of the cases a multiply connected Fatou component of a function with at least one omitted value ultimately lands on a Herman ring of period at least 2. The current article proves that the existence of Herman rings is severely restricted whenever there is an omitted value and prove the following. Functions all of whose poles are multiple have no Herman ring. No Herman ring with period one or two exist. Functions with a pole which is an omitted value do not have any Herman ring. If a function has only a single pole then it has no Herman ring.

Is every doubly connected periodic Fatou component (of a meromorphic map) a Herman ring? The answer is yes when the period is one and this question is stated to be open for other periods [4]. We settle this question for all maps in M_o by proving that every doubly connected periodic Fatou component of such a map is a Herman ring.

Using quasi-conformal maps, Fagella et al. investigated Herman rings of transcendental maps and proved that a general meromorphic function having n poles cannot have more than n invariant Herman rings [6]. The same tool is exploited by Zheng for showing that a function of finite type has at most finitely many Herman rings [9]. Though we establish similar restrictions on Herman rings for functions in M_o, our approach does not use quasi-conformal maps and is mostly elementary.

In Section 2, a number of lemmas are proved that are required for the proofs later. Also an easy theorem establishing non existence of Herman rings whenever all poles are multiple is proved in this section. We provide all other results and their proofs in Section 3.

For a closed curve γ in $\mathbb{C}$, let $B(\gamma)$ denote the union of all the bounded components of $\widehat{\mathbb{C}} \setminus \gamma$. For a doubly connected domain H, $B(H)$ means the bounded component of $\widehat{\mathbb{C}} \setminus H$ and $\widetilde{H}$ denotes $H \bigcup B(H)$. The boundary and the closure of a domain D in $\widehat{\mathbb{C}}$ is denoted by ∂D and $\overline{D}$ respectively. For a Fatou component V of f, we denote the Fatou component containing $f^n(V)$ by V_n for $n = 0, 1, 2, ...$ where f^0 denotes the identity map. Denote the component of the Julia set $\mathcal{J}(f)$ containing a set A by J_A. Also O_f stands for the set of all omitted values of f. By ring, we shall mean Herman ring throughout.

2. Preliminary lemmas

A non-contractible Jordan curve in the Fatou set of $f \in M$ ultimately (under forward iteration) surrounds a pole and in the next iteration, all the omitted values whenever such values exist. This elementary but useful fact is already proved in [7], which we state as a lemma.

Lemma 2.1. *Let $f \in M$ and V be a multiply connected Fatou component of f. Also suppose that γ is a closed curve in V with $B(\gamma) \bigcap \mathcal{J}(f) \neq \emptyset$. Then there is an $n \in \mathbb{N} \bigcup \{0\}$ and a closed curve $\gamma_n \subseteq f^n(\gamma)$ in V_n such that $B(\gamma_n)$ contains a pole of f. Further, if $O_f \neq \emptyset$ then $O_f \subset B(\gamma_{n+1})$ for some closed curve γ_{n+1} contained in $f(\gamma_n)$.*

We say a Herman ring H encloses a point w if $B(H)$ contains w.

Remark 2.2.

1. The proof of the above lemma also gives that $O_f \bigcap \overline{f(B)} = \emptyset$ for every bounded domain B.
2. If H is a Herman ring of $f \in M_o$ then some H_j encloses a pole and its forward image H_{j+1} encloses O_f. Consequently, the Julia component containing such a pole or an omitted value is always bounded.

We are to make a the above result more precise for which the following definition is required.

Definition 2.3. (**Simply bounded pole**)
A pole of a function is called simply bounded if it is simple and the component (maximally connected subset) of the Julia set containing it, is bounded and does not contain any other pole of the function.

Lemma 2.4. *If H is a Herman ring of $f \in M_o$ then $f : B(H) \to \widehat{\mathbb{C}}$ is one-one.*

Proof. Let H be a p-periodic Herman ring and γ be an f^p-invariant Jordan curve in it. Since $f : H \to \mathbb{C}$ is one-one by the definition of Herman ring, $f(\gamma)$ is a Jordan curve winding around every point of $B(H_1)$ exactly once.

If $f : B(H) \to \widehat{\mathbb{C}}$ is not one-one then $f : B(\gamma) \to \widehat{\mathbb{C}}$ is not one-one.

Let $f : B(\gamma) \to \widehat{\mathbb{C}}$ be analytic. Then there are at least two points in $B(H)$ with the same image and by the Argument principle, the curve $f(\gamma)$ winds around every point of $B(H_1)$ at least twice. If $f : B(\gamma) \to \widehat{\mathbb{C}}$ has at least a pole then Lemma 2.1 ensures that $f(\gamma)$ encloses an omitted value a of f. Applying the Argument principle to $g(z) = f(z) - a$ on $B(\gamma)$, we find $g(\gamma)$ winding around 0 at least twice implying that $f(\gamma)$ winds around a at least twice. This is a contradiction.

□

Remark 2.5. Above lemma indeed gives that every pole enclosed by a Herman ring is simply bounded and a Herman ring can enclose at most one pole.

Above lemma along with Lemma 2.1 gives the following result.

Theorem 2.6. *If all the poles of a function in M_o are multiple then it has no Herman ring.*

Given a Herman ring H of period p, let γ be an f^p-invariant Jordan curve in it and set $\gamma_j = f^j(\gamma)$ for $j = 0, 1, 2, ..., p-1$. Each γ_j is bounded and their arrangement in the plane is precisely that of the periodic cycle of Herman rings containing H.
Define
$$\mathcal{K}(H) = \widehat{\mathbb{C}} \setminus \bigcup_{j=0}^{p-1} \gamma_j.$$
We simply write $\mathcal{K}$ instead of $\mathcal{K}(H)$ whenever H is understood from the context.

Note that for every H, $\mathcal{K}(H)$ has exactly one unbounded component and at least one simply connected bounded component. Further, each of its components intersects the Julia set. In this context, the choice of γ is immaterial as long as it is f^p- invariant and is in H. Given a Herman ring H, we first make three definitions.

Definition 2.7. (**H-permissible number**)
A natural number n is called H-permissible if there is an n-connected component of $\mathcal{K}(H)$.

It is important to note that the number of H-permissible numbers cannot exceed the number of components of $\mathcal{K}$.

Definition 2.8. (**H-relevant pole**)
A pole w is called H-relevant if some H_j encloses w.

H-relevant poles are always simply bounded by Remark 2.5.

Definition 2.9. (**H-Maximal nest**)
An H-maximal nest is a collection of (Herman) rings belonging to the periodic cycle containing H such that one of these rings, called the outermost ring, encloses all other rings of the collection.

Note that the number of H-maximal nests can be any natural number less that or equal to the period of H. The unbounded component of $\mathcal{K}$ is n-connected if and only if there are n many H-maximal nests. Further, each such nest has at least one innermost ring and exactly one outermost ring.

We are now to relate H-permissible numbers, H-relevant poles and H-maximal nests.

Lemma 2.10. *Let H be a Herman ring of a function $f \in M_o$. If C, P and N denote the set of all H-permissible numbers, of all H-relevant poles and of all the H-maximal nests respectively then $|C| \leq |P| \leq |N|$ where $|.|$ denotes the number of elements in a set.*

Proof. Let $n \in C$ and K_n be a component of $\mathcal{K}$ with connectivity n. Since $K_n \bigcap \mathcal{J}(f) \neq \emptyset$ and the boundary of K_n is in the Fatou set, we can find the smallest non-negative natural number m such that $f^m(K_n)$, which we denote by K_n^m, contains a pole w_n. This pole is clearly H-relevant and is the only pole in K_n^m. Define $h : C \to P$ by $h(n) = w_n$. Clearly, the set K_n^m is an n-connected component of $\mathcal{K}$ by Lemma 2.4. For two distinct n_1, n_2 in C, the connectivities of $K_{m_1}^{n_1}$ and $K_{m_2}^{n_2}$ are different where m_i is the smallest non-negative integer corresponding to n_i for $i = 1, 2$, as above. Therefore $K_{n_1}^{m_1}$ and $K_{n_2}^{m_2}$ are two different components of $\mathcal{K}$ which means that they are disjoint. The poles they contain are different from each another. Thus, the function h is one-one and $|C| \leq |P|$. No two H-relevant poles can be enclosed by a Herman ring belonging to a single H-maximal nest by Remark 2.5. Also, each H-relevant pole is in a maximal nest. Therefore $|P| \leq |N|$ completing the proof. □

Remark 2.11.

1. Two components of $\mathcal{K}(H)$ with the same connectivity may correspond to a single H-relevant pole.
2. One may find a situation in which the map h shall be onto.
3. For each H-permissible n, there is an n-connected component of $\mathcal{K}(H)$ containing a pole. This is clear from the above proof.

Every component of the pre-image of all sufficiently small neighbourhood of an omitted value is unbounded and on each of these components, the map is not one-one. Now, it follows from the definition (of Herman ring) that a Herman ring does not contain any omitted value. Each periodic cycle of Herman rings of a function in M_o contains a rings enclosing the set of all omitted values as well as a ring enclosing a pole by Lemma 2.1. These two rings may be the same. Since invariant Herman rings do not exist for these functions [7], locating the omitted values and the poles in different components of $\mathcal{K}(H)$ for a Herman ring H requires some work.

Given a p-periodic Herman ring $H = H_0$ and a non empty set A contained in a bounded component of $\mathcal{K}(H)$ such that $A \bigcap (\bigcup_{i=0}^{p-1} H_i) = \emptyset$, we say H_j is the innermost ring with respect to A if there is no $H_k, k \neq j$ enclosing A and enclosed by H_j.

Lemma 2.12. *Let H be a p-periodic Herman ring of $f \in M_o$. Then we have the following.*

(1) *The outermost ring of each H-maximal nest encloses at most one pole.*
(2) *The set O_f is contained in a single bounded component of $\mathcal{K}$.*
(3) *Either the innermost ring $H(in)$ with respect to O_f does not enclose any pole or is an innermost ring of the H-maximal nest containing it.*
(4) *If only one H-maximal nest N has more than one ring in it then an innermost ring of N encloses O_f.*

Proof. (1) That the outermost ring of each H-maximal nest encloses at most one pole follows from Remark 2.5.

(2) We assume $|O_f| = 2$ otherwise there is nothing to prove. Suppose that O_f intersects two distinct components of $\mathcal{K}$. Then there is a Herman ring, say H_j enclosing exactly one omitted value of f. If H_{j-1} is the Herman ring such that $f(H_{j-1}) = H_j$ then consider an f^p-invariant Jordan curve γ in H_{j-1}. It now follows that f is not analytic on $B(\gamma)$ by Remark 2.2 and hence not on $B(H_{j-1})$. Therefore $B(H_{j-1})$ contains a pole and hence H_j encloses both the omitted values by Lemma 2.1. This contradicts our assumption proving that O_f is contained in a single component of $\mathcal{K}$.

(3) If the innermost ring $H(in)$ with respect to O_f encloses a pole then $f^n(H(in))$ encloses or is equal to $H(in)$ for all n by Lemma 2.1. That means $H(in)$ is the innermost ring of the H-maximal nest containing it.

(4) If N is the only H-maximal nest containing more than one ring in it then there is a H-permissible number other than 1. By Remark 2.11(3), there is a pole w enclosed by a ring of N which is different from all its innermost rings. No innermost ring of this nest encloses a pole by Remark 2.5. But by Lemma 2.10, the innermost ring of some maximal nest other than N encloses a pole. Now, there are at least two rings enclosing poles and their forward images (which are distinct) must enclose O_f. Since N is the only H-maximal nest having more than one ring, these forward images must be in N.

If O_f is not enclosed by an innermost ring of N then the innermost ring with respect to O_f, say H_k is different from all the innermost rings of N. Further H_k encloses a ring $H_{k'}$ of N. By (3) of this lemma, H_k encloses a pole. But in this situation, $f^n(H_k)$ is different from $H_{k'}$ for all n by Lemma 2.1 which is a contradiction. Thus, all the omitted values are enclosed by an innermost ring of N.

□

Now, the number of rings in a periodic cycle of Herman rings that enclose some pole is to be determined. An H-maximal nest $\{H_{i_1},\ H_{i_2},\ H_{i_3},...,\ H_{i_n}\}$ is called strictly nested if for all $1 \leq j \neq k \leq n$, either $H_{i_j} \subset B(H_{i_k})$ or $H_{i_k} \subset B(H_{i_j})$.

Lemma 2.13. *For every p-periodic Herman ring H of $f \in M_o$, the number of elements in the set $\{0 \leq i \leq p\ :\ H_i$ encloses a pole$\}$ is even.*

Proof. Let H_j be a p-periodic Herman ring and γ_1, γ_2 be two f^p-invariant Jordan curves in H_j such that $\gamma_1 \subset B(\gamma_2)$. If H_j encloses a pole then $f(\gamma_2) \subset B(f(\gamma_1))$. Otherwise $f(\gamma_1) \subset B(f(\gamma_2))$. This is true for every $j \geq 0$ by Lemma 2.1 and 2.4. If the number of Herman rings in this cycle that encloses some pole is odd then $f^p(\gamma_1)$ encloses $f^p(\gamma_2)$ which is a contradiction as $f^p(\gamma_i) = \gamma_i$ for $i = 1, 2$. □

3. Results and their proofs

3.1. **Arrangement of Herman rings.** The arrangement in the plane of all the Herman rings of a p-periodic cycle is determined by $\mathcal{K}$. One way to broadly classify the possible arrangements may be in terms of the number of maximal nests which can be anything between 0 and $p+1$. However, the two extreme arrangements, namely when the number of maximal nests is 1 or p, turn out to be impossible. We first make a definition.

Definition 3.1. (**Nested, Strictly nested and Strictly non-nested**) A Herman ring H of period at least two is called *nested* if there is a j such that $H_i \subset B(H_j)$ for all $i \neq j$. It is called *strictly nested* if for each $i \neq j$, either $H_i \subset B(H_j)$ or $H_j \subset B(H_i)$. We say H is *strictly non-nested* if $B(H_i) \bigcap B(H_j) = \emptyset$ for all $i \neq j$.

If a Herman ring is nested or strictly non-nested then by definition, so are all the Herman rings in the cycle containing it. Each Herman ring of period two is either nested or strictly non-nested.

It is shown in [7] that functions having only one pole has no Herman ring with period 2. That Herman rings of period two also do not exist for any function in M_o (possibly with more than one pole) is the content of the following result.

Theorem 3.2. *If $f \in M_o$ then f has no Herman ring which is nested or strictly non-nested and in particular, it has no Herman ring of period one or two. Further, if a pole of f is an omitted value then it has no Herman ring of any period.*

Proof. Let H_0 be a p-periodic Herman ring. A function in M_o has no invariant Herman ring [7] for which we can assume $p > 1$. The proof will follow by deriving contradictions in each of the following cases.

Case I: H is nested

Without loss of generality suppose that $H_i \subset B(H_0)$ for all $i > 0$. Then by Lemma 2.1, there is a pole of f in $B(H_0)$ and all the omitted values are in $B(H_1)$. Further, the set $f(B(H_0))$ is the unbounded component of $\widehat{\mathbb{C}} \setminus H_1$ by Lemma 2.4. Since $\bigcup_{i=1}^{p-1} H_i \subset B(H_0)$, we must have $\bigcup_{i=2}^{p-1} H_i \bigcup H_0 \subset f(B(H_0)) = \widehat{\mathbb{C}} \setminus (H_1 \bigcup B(H_1))$. Then H_1 is inner in the sense that $B(H_1)$ does not contain

H_j for any $j \geq 0$. If $B(H_1)$ contains a component of $f^{-1}(H_j)$ for some j then this component is not periodic and is in $B(H_0)$ (since $H_1 \subset B(H_0)$). Also, there is a periodic Herman ring mapped onto H_j by f which is either H_0 or in $B(H_0)$. Both these periodic and non-periodic components are mapped onto H_j and are in $B(H_0)$ giving that $f : B(H_0) \to \widehat{\mathbb{C}}$ is not one-one: a contradiction to Lemma 2.4. Thus $B(H_1)$ does not contain any component of $f^{-1}(H_j)$ for any j and therefore $f(B(H_1))$ contains no H_j for any $j \geq 0$. Now, if f has a pole in $B(H_1)$ then $H_2 = H_0$. Otherwise, that means if f is analytic in $B(H_1)$ then H_2 is inner. Repeating this argument and noting that $B(H_1)$ intersects the Julia set and hence, contains pre-images of poles, the smallest natural number j^* can be found such that H_j is inner for all $j, 0 < j < j^*$, H_{j*-1} encloses a pole and $H_{j^*} = H_0$. This means that the forward image of every inner ring is inner or H_0 and that of the H_0 is inner. Now, take an f^p-invariant Jordan curve γ_0 in H_0 and consider the region A bounded by $\gamma = \bigcup_{i=0}^{j^*-1} f^i(\gamma_0)$. Any pole in A would violate the univalence of f in $B(H_0)$, since there is a pole in $B(H_{j^*-1}) \subset B(H_0)$. Thus f is conformal in A by Lemma 2.4. Further $f(A) = A$ because $f(\partial A) = f(\gamma) = \gamma$ which gives that $f^n(A) = A$ for all n. But this is not possible as A intersects the Julia set.

Case II: H is strictly non-nested

Suppose that H_0 is strictly non-nested and without loss of generality that $B(H_0)$ contains a pole of f. Then $B(H_1)$ contains O_f by Lemma 2.1 and by Lemma 2.13, there is a ring H_i, $i > 1$ containing a pole. Therefore $O_f \subset B(H_{i+1})$. But this gives that $B(H_1) \subseteq B(H_{i+1})$ or $B(H_{i+1}) \subset B(H_1)$ which is not possible as H_0 is strictly non-nested.

If there is a Herman ring of period two then it is nested or strictly non-nested. Similarly, if a pole of f is an omitted value then the forward images of each ring, enclosing this pole, also encloses the pole. That means the Herman ring is nested. But these kinds of Herman rings are not possible proving the theorem. □

3.2. **Functions with a single pole.** It is already proved that if a function $f \in M_o$ has a pole which is also an omitted value then it cannot have any Herman ring. Here we look at the possibility of Herman rings for functions with only a single pole which is not necessarily an omitted value. If the pole is not simply bounded then Herman rings do not exist by Remark 2.5. We assume that the pole is simply bounded and denote it by w. Before presenting the main result of this section, we first state and prove two elementary lemmas.

We say, a maximal nest of Herman rings is strictly nested if the complement of their union has exactly one bounded simply connected domain. In this case, there is a unique ring enclosed by all other in the nest. Such a ring is called the innermost ring of N. By $|N|$, we mean the number of rings in the maximal nest N.

Lemma 3.3. *For a Herman ring H, let there be only two H-maximal nests N_1 and N_2, each of which is strictly nested. If a ring of N_1 encloses O_f then $|N_1| > |N_2|$.*

Proof. Without loss of generality, suppose that $H \in N_1$ is the innermost ring with respect to O_f. If H encloses a pole then it will be nested by Lemma 2.1 which cannot be true in view of Theorem 3.2. Let H_{in} be the innermost ring of N_1. Since

N_1 is strictly nested, either $H = H_{in}$ or $H_{in} \subset H$. In both the cases, H_{in} does not enclose any pole. This along with Remark 2.11(3) gives that the innermost ring of N_2 encloses a pole and by Lemma 2.1, the forward images of all the rings belonging to N_2 encloses O_f. There is also a pole enclosed by some ring of N_1 by Lemma 2.10. The forward image of this ring is different from the forward images of all rings belonging to N_2. This means that the number of rings enclosing O_f is more than $|N_2|$. Therefore $|N_1| > |N_2|$. $\square$

As discussed in the proof of Theorem 3.2, a Herman ring may have many pre-images out of which only one will be periodic and we call that as periodic pre-image of the ring.

Lemma 3.4. *Let $f \in M_o$ have only one simply bounded pole w. If H_m is the outermost ring with respect to w then $B(H_m) \bigcap O_f = \emptyset$ and H_m does not enclose any pre-image of w.*

Proof. If $B(H_m) \bigcap O_f \neq \emptyset$ then the periodic pre-image H_{m-1} of H_m must enclose a pole by Remark 2.2. This pole is w as this is the only simply bounded pole of f. Note that O_f is enclosed by the innermost ring with respect to w by Lemma 2.12(3). Since the periodic pre-image H_{m-1} of H_m encloses w, H_{m-1} also encloses O_f. Similarly, it follows that the periodic pre-image H_{m-2} of H_{m-1} encloses w as well as O_f. By repeating this process for finitely many times, it is seen that H_j encloses w for all $j \geq 0$. In other words, H_m is nested which is not possible by Theorem 3.2.

If H_m encloses a pre-image w_{-1} of w then H_{m+1} encloses w as well as O_f and therefore $B(H_m) \bigcap O_f \neq \emptyset$. This is already shown to be impossible. $\square$

Remark 3.5. Above Lemma implies that both the forward image and the periodic pre-image of any H_m is in nests different from the one enclosing w.

Now we present the main result, an easy-to-verify criteria for non-existence of Herman rings for functions with a single pole.

Theorem 3.6. *If $f \in M_o$ has only one pole then f has no Herman ring.*

Proof. Suppose that H is a p-periodic Herman ring of f. Then some H_i encloses a pole, say w by Lemma 2.1. Let all such Herman rings be enumerated as $\{H_{i_j}\}_{j=1}^{n}$. Also, let H_{i_1} be the innermost and H_{i_n} be the outermost ring with respect to w. Note that each H_{i_j+1} encloses O_f by Lemma 2.1. By Lemma 3.4, none of the omitted values lie in $B(H_{i_n})$. Therefore, each of $\{H_{i_j+1}\}_{j=1}^{n}$ is in a maximal nest different from the one containing H_{i_j}s. Let m be the least natural number such that H_{i_n+1+m} encloses a pole. This pole is none other than w by our assumption. Consequently $H_{i_n+1+m} = H_{i_j}$ for some j. This j can only be 1, otherwise there will be at least $n+1$ Herman rings enclosing w which is not true. Thus $H_{i_n+1+m} = H_{i_1}$ and consequently $H_{i_1+1+m} = H_{i_n}$. Note that $p = 2(m+1)$. Consider a non-contractible f^p-invariant Jordan curve γ_1 in H_{i_1}. Then $\gamma_{m+1} = f^{m+1}(\gamma_1)$ is also a non-contractible Jordan curve in H_{i_n} and $f^{m+1}(\gamma_{m+1}) = \gamma_1$. The map f^{m+1} takes the region bounded by these two curves γ_{m+1} and γ_1 conformally onto itself. This is not possible if $n > 1$ as this region intersects the Julia set. But by Lemma 2.13, $n = 1$ is not possible. This contradiction completes the proof. $\square$

Remark 3.7. If a function in M_o has more than one pole but only one simply bounded pole then it does not have any Herman ring.

Example 3.8. For each entire map g and non-zero complex number z_0, the map $f(z) = \frac{e^{g(z)}}{(z-z_0)^k}$ is meromorphic with only a single pole z_0 which is different from its omitted value. By the above theorem, it has no Herman ring.

Here is a result that follows from the arguments used in Lemma 3.4 and Lemma 2.13.

Theorem 3.9. *For a Herman ring H of a function in M_o, if there are only two H-maximal nests, one of which consists of only one ring and the other is strictly nested, then H is odd periodic.*

Proof. If there are only two H-maximal nests and one of them consists of only one ring then the other, we call it bigger, encloses O_f in one of its innermost ring by Lemma 2.12(4). Further, there is a pole w enclosed by the outermost ring of the bigger nest by Lemma 2.10. Suppose that this pole and O_f are separated by at least two rings, say H^1 and H^2. Each ring of this nest enclosing w also encloses O_f (Otherwise H will be nested by the arguments used in the proof of Lemma 3.4) and thus its periodic pre-image encloses a pole other than w. If k is the smallest natural number such that H_k^1 and H_k^2 encloses w then their periodic pre-images H_{k-1}^1 and H_{k-1}^2 must be in a H-maximal nest which is different from the bigger nest and has more than one ring: not possible. Therefore, the pole w and O_f are separated by only one ring of the bigger nest. All rings of the bigger nest except the innermost encloses w and the single ring of the other nest also encloses a pole making the total number of rings enclosing some pole even. Thus there are odd number of rings in the cycle containing H proving that H is odd periodic.

□

3.3. **Doubly connected Fatou components.** Herman ring is a periodic doubly connected Fatou component. But the converse is not at all obvious. It is an open question that whether a doubly connected periodic Fatou component of a meromorphic function (with a single essential singularity) is always a Herman ring [4]. This is known to be true when period is one [1]. We prove it for every period but only for functions in M_o. For proving this, we put together Theorems 1-5 of [7] as a lemma. We say a Fatou component V is SCH if one of the following holds.

1. V is simply connected.
2. V is multiply connected with $c(V_n) > 1$ for all $n \in \mathbb{N}$ and $V_{\bar{n}}$ is a Herman ring for some $\bar{n} \in \mathbb{N} \bigcup \{0\}$.

Let $M_o^k = \{f \in M_o \ : \ f \text{ has } k \text{ omitted values}\}$ for $k = 1, \ 2$.

Lemma 3.10. *Let $f \in M_o$.*

1. *Let $\mathcal{J}(f) \bigcap O_f \neq \emptyset$. If $f \in M_o^2$ or $f \in M_o^1$ with $|\mathcal{J}_{O_f}| > 1$, then each Fatou component of f is SCH.*
2. *Let the set O_f intersect two distinct Fatou components U_1 and U_2 of f. If both U_1 and U_2 are unbounded, or exactly one of them is unbounded and is*

simply connected then all the Fatou components of f are simply connected. Otherwise, each Fatou component of f is SCH.

(3) *Let O_f be contained in a Fatou component U and V be a Fatou component with $V_n \neq U$ for any $n \geq 0$.*
   (a) *If U is unbounded, then $c(V_n) = 1$ for all $n \geq 0$.*
   (b) *If U is bounded, then V is SCH.*
   (c) *If U is wandering, then $c(U_n) = 1$ for all $n \geq 0$.*
   (d) *Let U be pre-periodic but not periodic. If U is unbounded, then $c(U_n) = 1$ for all $n \geq 0$. If U is bounded, then U is SCH.*
   (e) *If U is periodic, then $c(U_n) = 1$ or ∞ for all $n \geq 0$.*

(4) *Let $f \in M_o^1$, $O_f = \{a\} \subset \mathcal{J}(f)$ and $|\mathcal{J}_a| = 1$. If $\mathcal{J}_a$ is not a buried component of the Julia set, then f has an infinitely connected Baker domain B with period $p > 1$ and a is a pre-pole. Further, for each multiply connected Fatou component U of f not landing on any Herman ring, there is a non-negative integer n depending on U such that $U_n = B$. In this case, singleton buried components are dense in $\mathcal{J}(f)$.*

(5) *Let $f \in M_o^1$, $O_f = \{a\} \subset \mathcal{J}(f)$ and $|\mathcal{J}_a| = 1$. If $\mathcal{J}_a$ is a buried component of the Julia set, then all the multiply connected Fatou components not landing on any Herman ring are wandering and a is a limit point of $\{f^n\}_{n>0}$ on each of these wandering domains. Further, if $\mathcal{F}(f)$ has a multiply connected wandering domain, then the forward orbit of a is an infinite set and singleton buried components are dense in $\mathcal{J}(f)$.*

Theorem 3.11. *For every $f \in M_o$, each doubly connected periodic Fatou component is a Herman ring.*

Proof. Let D be a doubly connected periodic Fatou component of $f \in M_o$.

Let $\mathcal{J}(f) \bigcap O_f \neq \emptyset$. If (1)$f \in M_o^2$, (2) $f \in M_o^1$ with $|\mathcal{J}_{O_f}| > 1$, or (3) $f \in M_o^1$, $O_f = \{a\} \subset \mathcal{J}(f)$, $|\mathcal{J}_a| = 1$ and $\mathcal{J}_a$ is a buried component of the Julia set then D is a Herman ring by Lemma 3.10(1) and (5). If $f \in M_o^1$, $O_f = \{a\} \subset \mathcal{J}(f)$, $|\mathcal{J}_a| = 1$ and $\mathcal{J}_a$ is not a buried component of the Julia set then the Baker domain (as mentioned in Lemma 3.10(4)) is infinitely connected. Indeed, it can be shown (see Lemma 4 and the proof of Theorem 4 of [7]) that all its forward images are also infinitely connected. Thus in this situation also, D is a Herman ring.

Let $\mathcal{J}(f) \bigcap O_f = \emptyset$. If O_f intersects two Fatou components then we are done by Lemma 3.10(2). Let $O_f \subset U$ for some Fatou component. Then, by Lemma 3.10(3)(a),(b), D is a Herman ring whenever $D_n \neq U$. If $D_n = U$ for some n then U must be periodic and by Lemma 3.10(3)(e), $c(D_n) = 1$ or ∞ for all n, which can not be possible as D is periodic and doubly connected. □

References

1. I. N. Baker, J. Kotus, and L. Yinian, *Iterates of meromorphic functions IV: Critically finite functions*, Results in Mathematics **22** (1992), 651–656.
2. I.N. Baker and P. Dominguez, *Analytic self-maps of the punctured plane*, Complex Variables and Elliptic Equations **37** (1998), no. 1, 67–91.

3. W. Bergweiler and A. E. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Revista Math. Iberoamericana **11** (1995), 355–373.
4. A. Bolsch, *Periodic Fatou components of meromorphic functions*, Bull. London Math. Soc. **31** (1999), 543–555.
5. P. Domínguez and N. Fagella,*Existence of Herman rings for meromorphic functions*, Complex Variables **49** (2004), 851–870.
6. N. Fagella and J. Peter,*On the configuration of Herman rings of meromorphic functions*, Journal of Mathematical Analysis and Applications **394** (2012), 458–467.
7. T. Nayak and J. H. Zheng, *Omitted values and dynamics of transcendental meromorphic functions*, J. London Math. Soc. **83** (2011), no. 1, 121–136.
8. Gong Z.M, Qiu W.Y, and Ren F.Y, *A negative answer to a problem of Bergweiler*, Complex variables Theory Appl. **30** (1996), 315–322.
9. J. Zheng, *Remarks on Herman rings of transcendental meromorphic functions*, Indian J. Pure Appl. Math.omplex variables Theory Appl. **31 (7)** (2000), 747–751.

School of Basic Sciences, Indian Institute of Technology Bhubaneswar, India
E-mail address: tnayak@iitbbs.ac.in